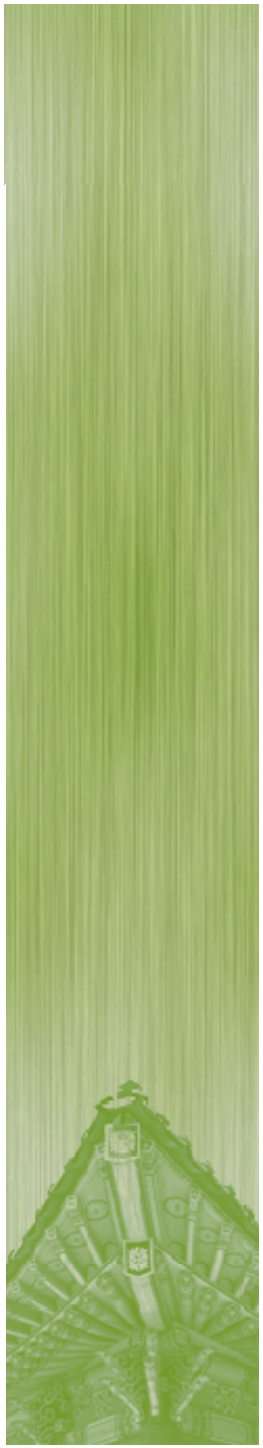

# Floating Point Arithmetic

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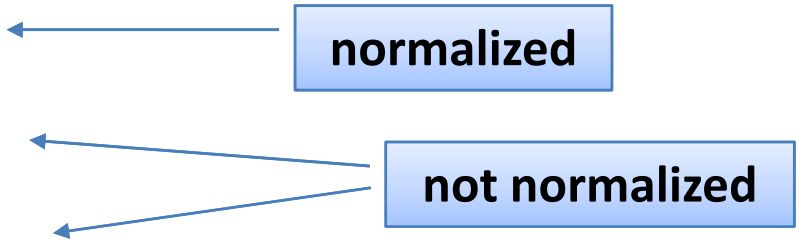
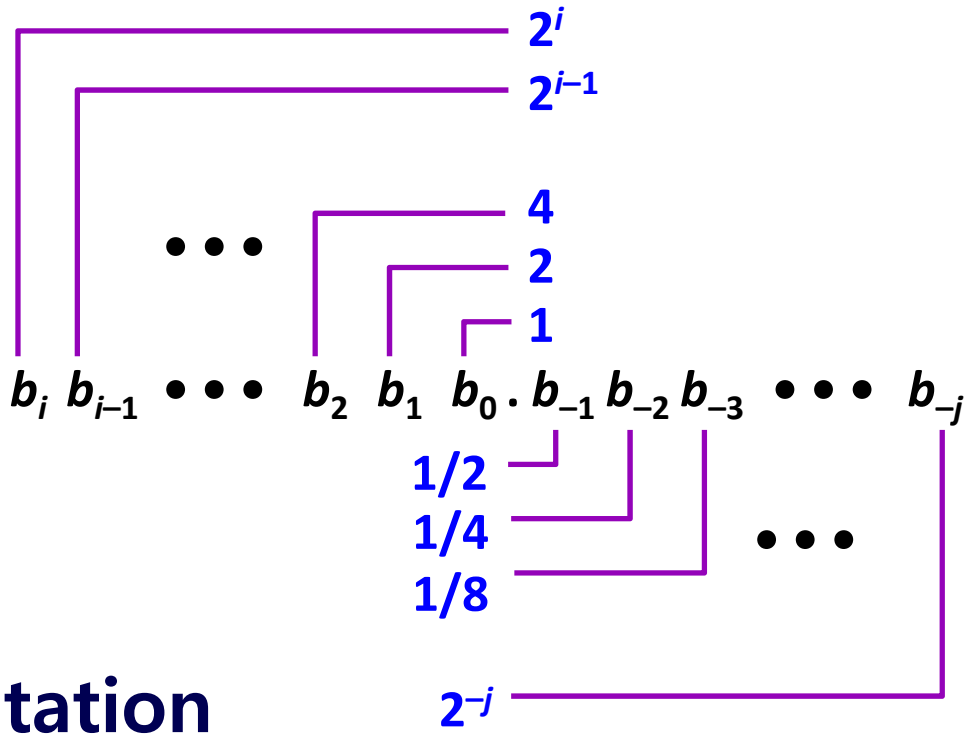
# Floating Point (1)

- **Representation for non-integral numbers**
  - Including very small and very large numbers
  - Types `float` and `double` in C
- **Like scientific notation**
  - $-2.34 \times 10^{56}$
  - $+0.002 \times 10^{-4}$
  - $+987.02 \times 10^{9}$




- **In binary**
  - $\pm 1.xxxxxxx_2 \times 2^{yyyy}$

# Floating Point (2)




- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

# Floating Point (3)

- **Floating point standard**
  - Defined by IEEE Std 754-1985
  - Developed in response to divergence of representations
    - Portability issues for scientific code
  - Now almost universally adopted
  - Two representations
    - Single precision (32-bit)
    - Double precision (64-bit)

# IEEE Floating Point (1)

- **Format**


$$x = (-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$$

- S: sign bit (0 ⇒ non-negative, 1 ⇒ negative)
- Normalize significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
  - Significand is Fraction with the "1." restored
- Exponent: actual exponent + Bias
  - Ensures exponent is unsigned
  - Single: Bias = 127, Double: Bias = 1023

# IEEE Floating Point (2)

- **Single-precision range**
  - Exponents 00000000 and 11111111 reserved
  - Smallest value
    - Exponent: 00000001
      $\Rightarrow$ actual exponent = 1 – 127 = -126
    - Fraction: 000…00 $\Rightarrow$ significand = 1.0
    - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$
  - Largest value
    - Exponent: 11111110
      $\Rightarrow$ actual exponent = 254 – 127 = +127
    - Fraction: 111..11 $\Rightarrow$ significand $\approx$ 2.0
    - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$

# IEEE Floating Point (3)

- **Double-precision range**
  - Exponents 0000…00 and 1111…11 reserved
  - Smallest value
    - Exponent: 00000000001
      ⇒ actual exponent = 1 – 1023= -1022
    - Fraction: 000…00 ⇒ significand = 1.0
    - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$
  - Largest value
    - Exponent: 11111111110
      ⇒ actual exponent = 2046 – 1023 = +1023
    - Fraction: 111..11 ⇒ significand ≈ 2.0
    - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

# IEEE Floating Point (4)

- **Relative precision**
  - All fraction bits are significant
  - Single: approx $2^{-23}$
    - Equivalent to $23 \times \log_{10}2 \approx 23 \times 0.3 \approx 6$ decimal digits of precision
  - Double: approx $2^{-52}$
    - Equivalent to $52 \times \log_{10}2 \approx 52 \times 0.3 \approx 16$ decimal digits of precision

# IEEE Floating Point (5)

- **Example: –0.75**
  - $-0.75 = (-1)^1 \times 1.1_2 \times 2^{-1}$
  - S = 1
  - Fraction = $1000\ldots00_2$
  - Exponent = –1 + Bias
    - Single: –1 + 127 = 126 = $01111110_2$
    - Double: –1 + 1023 = 1022 = $01111111110_2$

  - Single: 10111111010000…00
  - Double: 10111111111010000…00

# IEEE Floating Point (6)

- **Example: 11000000101000…00 (single)**
  - S = 1
  - Fraction = $01000\ldots00_2$
  - Exponent = $10000001_2$ = 129

  - $x = (-1)^1 \times (1 + 0.01_2) \times 2^{(129 - 127)}$
    $= (-1) \times 1.25 \times 2^2$
    $= -5.0$

# IEEE Floating Point (7)

- **Denormal numbers**
  - Exponent = 000…0 $\Rightarrow$ hidden bit is 0

$$x = (-1)^S \times (0 + \text{Fraction}) \times 2^{-\text{Bias}}$$

  - Smaller than normal numbers
    - Allow for gradual underflow, with diminishing precision
  - Denormal with fraction = 000…0

$$x = (-1)^S \times (0 + 0) \times 2^{-\text{Bias}} = \pm 0.0$$

**Two representations of 0.0!**

# IEEE Floating Point (8)

- **Infinities**
  - Exponent = 111..1, Fraction = 000..0
  - ±Infinity (∞)
  - Can be used in subsequent calculations, avoiding need for overflow check
- **NaNs**
  - Exponent = 111...1, Fraction ≠ 000...0
  - Not-a-Number (NaN)
  - Indicates illegal or undefined result
    - e.g., 0.0 / 0.0
  - Can be used in subsequent calculations

# Tiny FP Example (1)

- **8-bit floating point representation**
  - The sign bit is in the most significant bit
  - The next four bits are the **exp**, with a bias of 7
  - The last three bits are the **frac**

- **Same general form as IEEE format**
  - Normalized, denormalized
  - Representation of 0, NaN, infinity

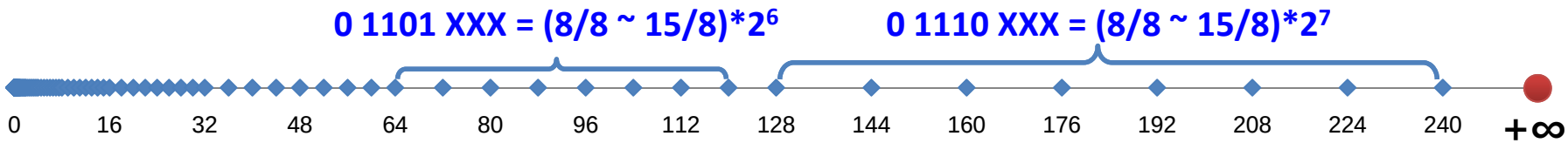
# Tiny FP Example (2)

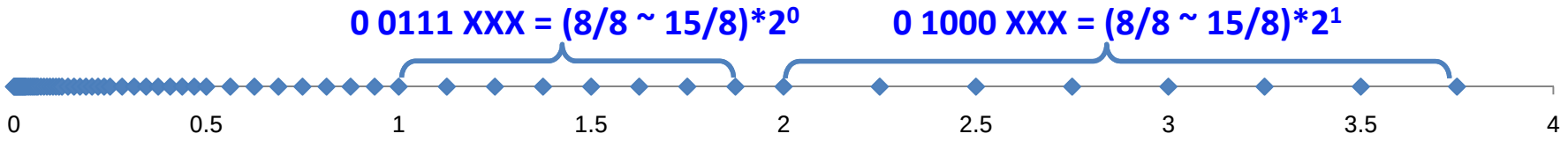
- Dynamic range

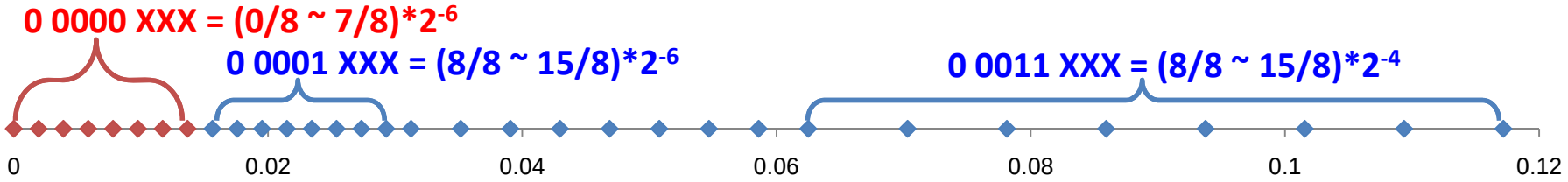
| Description | Bit representation | e | E | f | M | V |
|---|---|---|---|---|---|---|
| Zero | 0 0000 000 | 0 | -6 | 0 | 0 | 0 |
| Smallest pos. | 0 0000 001 | 0 | -6 | 1/8 | 1/8 | 1/512 |
| | 0 0000 010 | 0 | -6 | 2/8 | 2/8 | 2/512 |
| | 0 0000 011 | 0 | -6 | 3/8 | 3/8 | 3/512 |
| | 0 0000 110 | 0 | -6 | 6/8 | 6/8 | 6/512 |
| Largest denorm. | 0 0000 111 | 0 | -6 | 7/8 | 7/8 | 7/512 |
| Smallest norm. | 0 0001 000 | 1 | -6 | 0 | 8/8 | 8/512 |
| | 0 0001 001 | 1 | -6 | 1/8 | 9/8 | 9/512 |
| | 0 0110 110 | 6 | -1 | 6/8 | 14/8 | 14/16 |
| | 0 0110 111 | 6 | -1 | 7/8 | 15/8 | 15/16 |
| One | 0 0111 000 | 7 | 0 | 0 | 8/8 | 1 |
| | 0 0111 001 | 7 | 0 | 1/8 | 9/8 | 9/8 |
| | 0 0111 010 | 7 | 0 | 2/8 | 10/8 | 10/8 |
| | 0 1110 110 | 14 | 7 | 6/8 | 14/8 | 224 |
| Largest norm. | 0 1110 111 | 14 | 7 | 7/8 | 15/8 | 240 |
| Infinity | 0 1111 000 | - | - | - | - | +∞ |

# Tiny FP Example (3)

- Encoded values (nonnegative numbers only)




**(Without denormalization)**


**0 0000 XXX = (8/8 ~ 15/8)*2^-7**

# FP Addition (1)

- **4-digit decimal example:**

$$9.999 \times 10^{1} + 1.610 \times 10^{-1}$$

1. Align decimal points
   - Shift number with smaller exponent
   - $9.999 \times 10^{1} + 0.016 \times 10^{1}$
2. Add significands
   - $9.999 \times 10^{1} + 0.016 \times 10^{1} = 10.015 \times 10^{1}$
3. Normalize result & check for over/underflow
   - $1.0015 \times 10^{2}$
4. Round and renormalize if necessary
   - $1.002 \times 10^{2}$

# FP Addition (2)

- **4-digit binary example:**

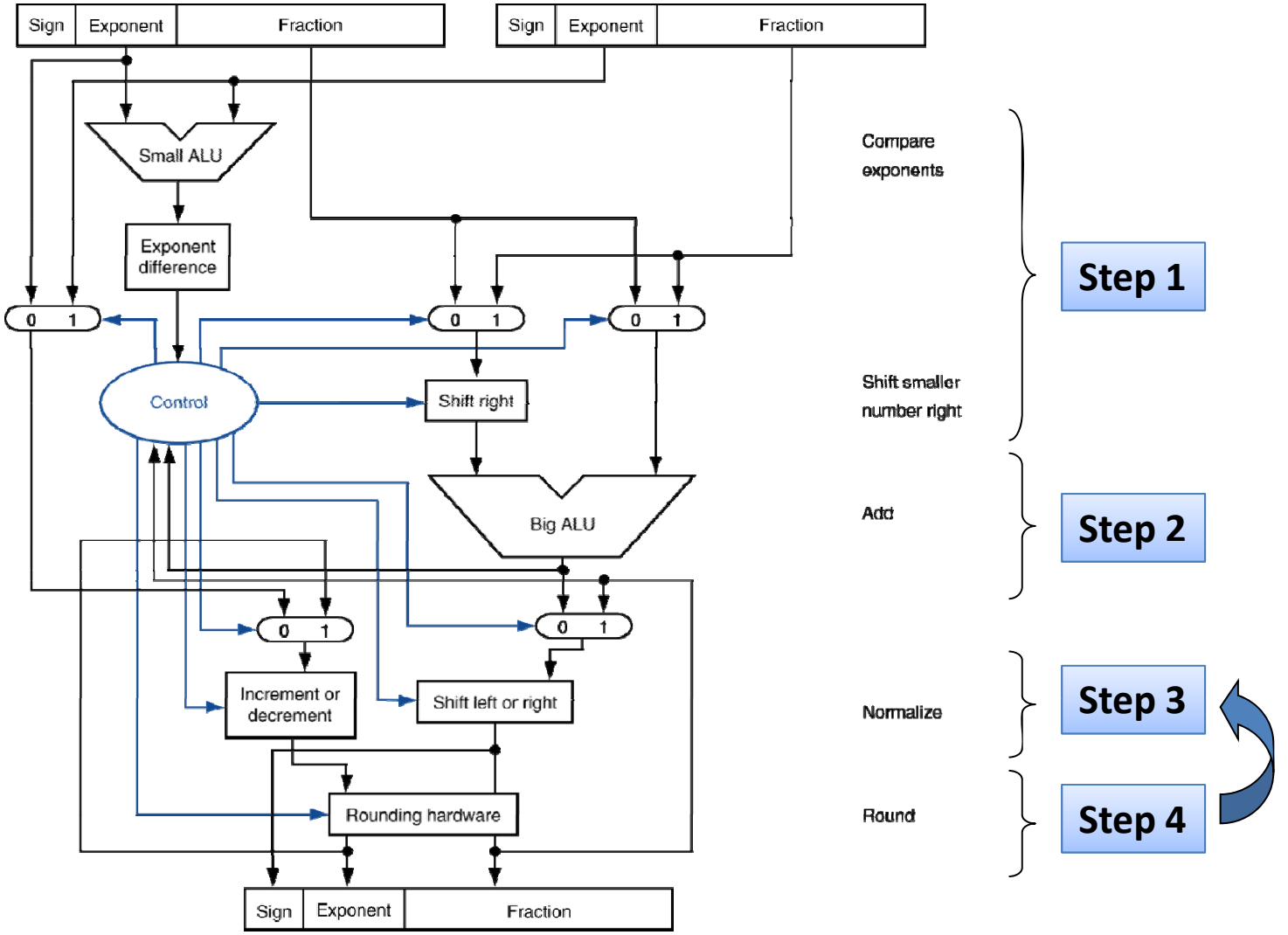
$$1.000_2 \times 2^{-1} + -1.110_2 \times 2^{-2} \quad (0.5 + -0.4375)$$

1. Align binary points
   - Shift number with smaller exponent
   - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1}$
2. Add significands
   - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-}1 = 0.001_2 \times 2^{-1}$
3. Normalize result & check for over/underflow
   - $1.000_2 \times 2^{-4}$, with no over/underflow
4. Round and renormalize if necessary
   - $1.000_2 \times 2^{-4}$ (no change) = 0.0625

# FP Addition (3)

- **FP adder hardware**
  - Much more complex than integer adder
  - Doing it in one clock cycle would take too long
    - Much longer than integer operations
    - Slower clock would penalize all instructions
  - FP adder usually takes several cycles
    - Can be pipelined

# FP Addition (4)



# FP Multiplication (1)

- **4-digit decimal example:**

$$1.110 \times 10^{10} \times 9.200 \times 10^{-5}$$

1. Add exponents
   - For biased exponents, subtract bias from sum
   - New exponent = 10 + –5 = 5
2. Multiply significands
   - $1.110 \times 9.200 = 10.212 \Rightarrow 10.212 \times 10^5$
3. Normalize result & check for over/underflow
   - $1.0212 \times 10^6$
4. Round and renormalize if necessary
   - $1.021 \times 10^6$
5. Determine sign of result from signs of operands
   - $+1.021 \times 10^6$

# FP Multiplication (2)

- **4-digit binary example:**

$1.000_2 \times 2^{-1} \times -1.110_2 \times 2^{-2}$ (0.5 × –0.4375)

1. Add exponents
   - Unbiased: –1 + –2 = –3
   - Biased: (–1+127) + (–2+127) = –3 + 254 – 127 = –3 + 127
2. Multiply significands
   - $1.000_2 \times 1.110_2 = 1.110_2 \Rightarrow 1.110_2 \times 2^{-3}$
3. Normalize result & check for over/underflow
   - $1.110_2 \times 2^{-3}$ (no change) with no over/underflow
4. Round and renormalize if necessary
   - $1.110_2 \times 2^{-3}$ (no change)
5. Determine sign: +ve × –ve ⇒ –ve
   - $-1.110_2 \times 2^{-3} = -0.21875$

# FP Instructions (1)

- **FP arithmetic hardware**
  - FP multiplier is of similar complexity to FP adder
    - But uses a multiplier for significands instead of an adder
  - FP arithmetic hardware usually does
    - Addition, subtraction, multiplication, division, reciprocal, square-root
    - FP ↔ integer conversion
  - Operations usually take several cycles
    - Can be pipelined

# FP Instructions (2)

- **FP instructions in MIPS**
  - FP hardware is coprocessor 1
    - Adjunct processor that extends the ISA
  - Separate FP registers
    - 32 single-precision: `$f0`, `$f1`, …, `$f31`
    - Paired for double-precision: `$f0/$f1`, `$f2/$f3`, …
    - Release 2 of MIPS ISA supports 32 × 64-bit FP registers
  - FP instructions operate only on FP registers
    - More registers with minimal code-size impact
  - FP load and store instructions
    - `lwc1`, `ldc1`, `swc1`, `sdc1`
    - e.g., `ldc1 $f8, 32($sp)`

# FP Instructions (3)

- **FP instructions in MIPS (cont'd)**
  - Single-precision arithmetic
    - `add.s, sub.s, mul.s, div.s`
    - e.g., `add.s $f0, $f1, $f6`
  - Double-precision arithmetic
    - `add.d, sub.d, mul.d, div.d`
    - e.g., `mul.d $f4, $f4, $f6`
  - Single- and double-precision comparison
    - `c.xx.s, c.xx.d (xx is eq, lt, le, …)`
    - e.g., `c.lt.s $f3, $f4`
  - Branch on FP condition code true or false
    - `bclt, bclf`
    - e.g., `bclt TargetLabel`

# FP Instructions (4)

- **Example: °F to °C**
  - C code:

```
float f2c (float fahr) {
   return ((5.0/9.0)*(fahr - 32.0));
}
```

    - fahr in $f12, result in $f0, literals in global memory space
  - Compiled MIPS code:

```
f2c: lwcl  $f16, const5($gp)
     lwc1  $f18, const9($gp)
     div.s $f16, $f16, $f18
     lwc1  $f18, const32($gp)
     sub.s $f18, $f12, $f18
     mul.s $f0, $f16, $f18
     jr    $ra
```

# Concluding Remarks

- **ISAs support arithmetic**
  - Signed and unsigned integers
  - Floating-point approximation to reals
- **Bounded range and precision**
  - Operations can overflow and underflow
- **MIPS ISA**
  - Core instructions: 54 most frequently used
    - 100% of SPECINT, 97% of SPECFP
  - Other instructions: less frequent