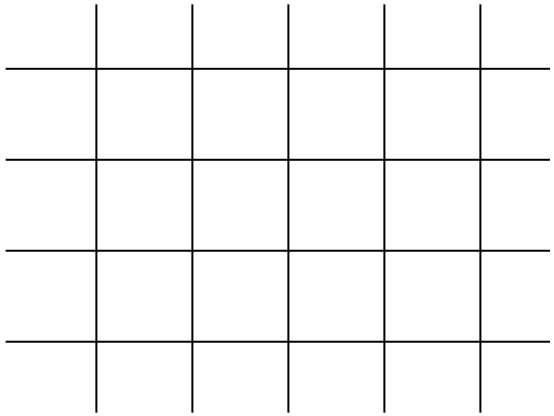
Grids
Geometry
Computational Geometry

Grids

- You will encounter many problems whose solutions need grid representation
  - map
  - VLSI layout
  - the most natural way to carve space into regions
- regular pattern
- rectangular or rectilinear grids are common
  - but sometimes you will see triangulr or hexagonal grids

Rectilinear Grids

- each cell is defined by horizontal and vertical lines
- spacing between lines can be
  - uniform
  - non-uniform
- three dimensional grid is difficult to visualize



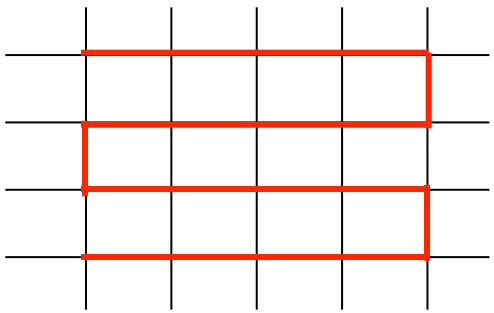
vertices

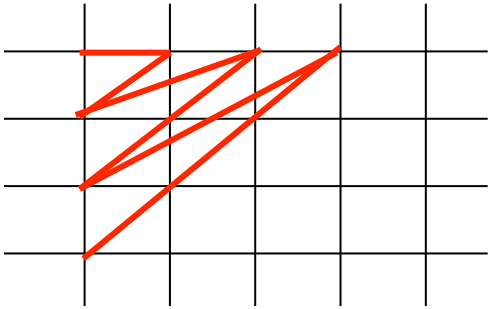
edges

cells

Graph Representation of Grid

- Grid is a graph (V, E)
  - each vertex contains information
    - city, village, transistor, ..
  - each edge represents relationship between vertices
  - can be 2D, 3D, 4D, ...
- Traversal
  - row major
  - column major
  - snake
  - diagonal
  - ....





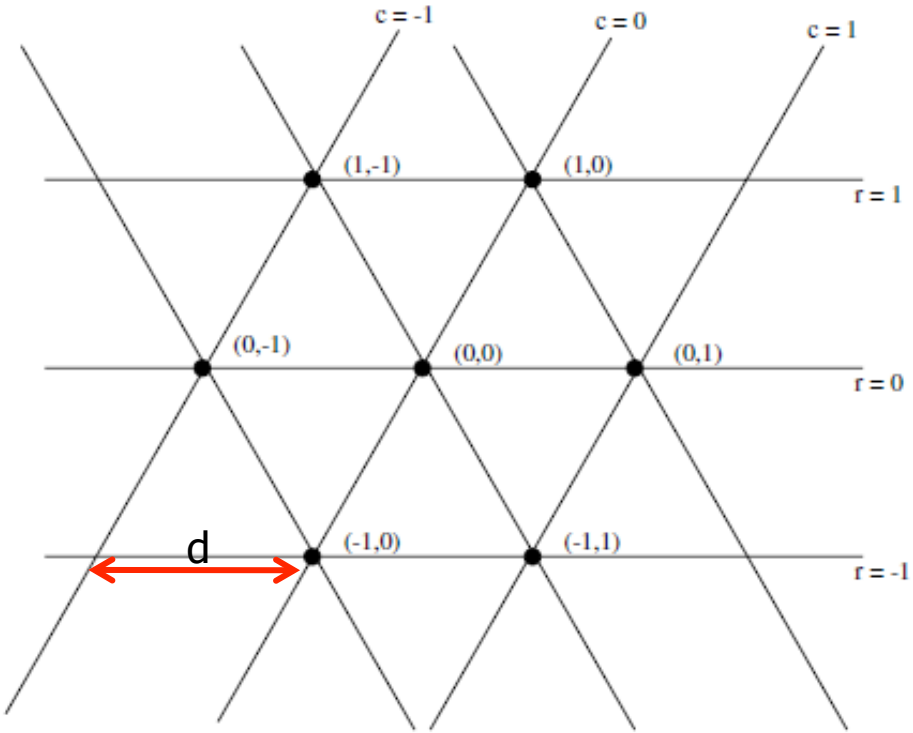
```
snake_order(int n, int m)
{
        int i,j;        /* counters */

        for (i=1; i<=n; i++)
            for (j=1; j<=m; j++)
                 process(i, j + (m+1-2*j) * ((i+1) % 2));
}
```

Dual Graphs

- m[i][j] can be either
  - a vertex
  - a face
  - Any one can be transformed into the other
  - proof?
- The four-color theorem
  - the map and the graph to solve the problem are different
- How can you represent weights on edges?

Triangular Lattices

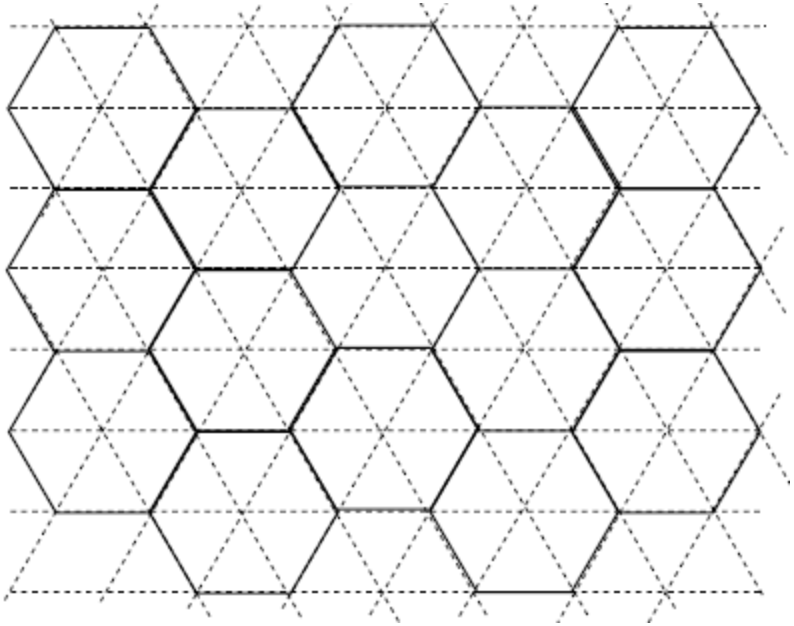


- a vertex needs to be assigned coordinates
  - (x, y) is good enough?
  - if edges are removed, it looks like a grid
- Duality?
- Geometrical coordinates

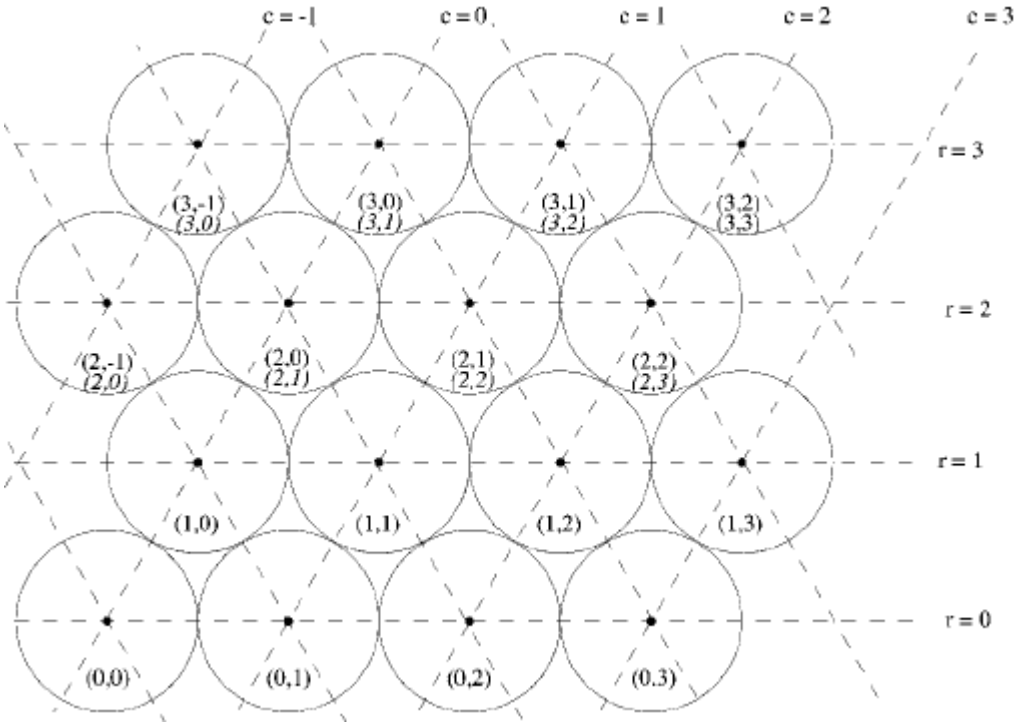
$$(x_g, y_g) = (d(x_t + (y_t \cos(60^o))), dy_t \sin(60^o))$$

Hexagonal Lattices

- It is a subset of a triangular lattice
  - remove every other vertex
- Interesting properties
  - it is the roundest polygon to fill the space
  - per space/perimeter
    - a circle is the best



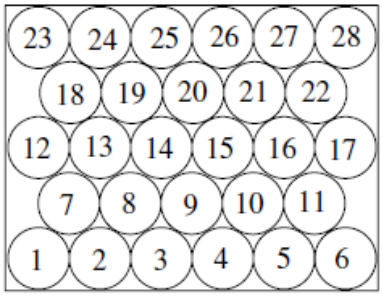
Representation

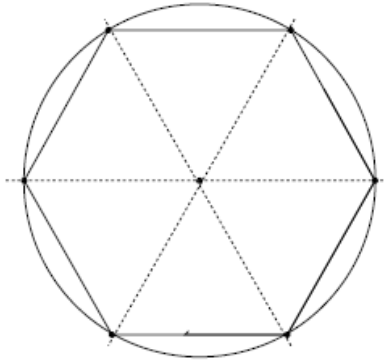


- for rectangular solution, additional coordinates are needed
  - (3,1) is better be at the straight above (1, 1)

Exercise

- Dish Packing
  - How many dishes can be places in a box?

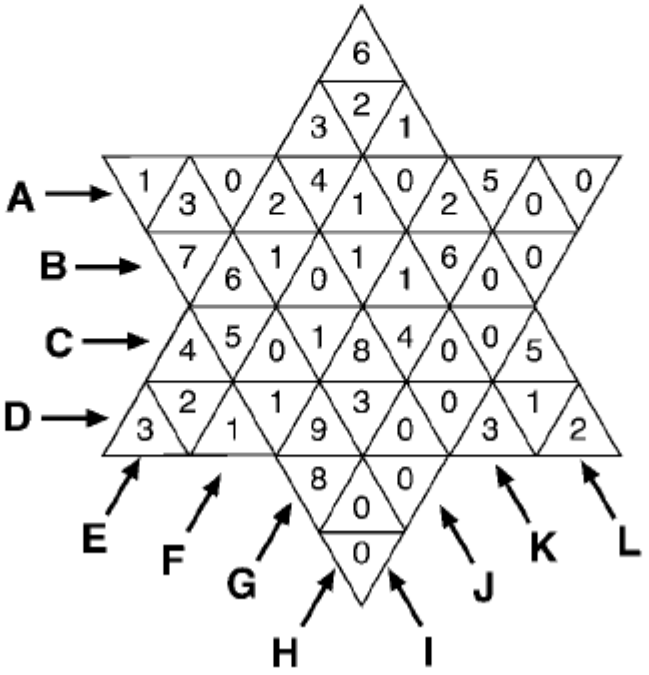




- Flying path
  - find the shortest flying distance from Seoul to Bagdad

Problem 1: Star

A board contains 48 triangular cells. In each cell is written a digit in a range from 0 through 9. Every cell belongs to two or three lines. These lines are marked by letters from A through L. See the figure below, where the cell containing digit 9 belongs to lines D, G, and I and the cell containing digit 7 belongs to lines B and I.



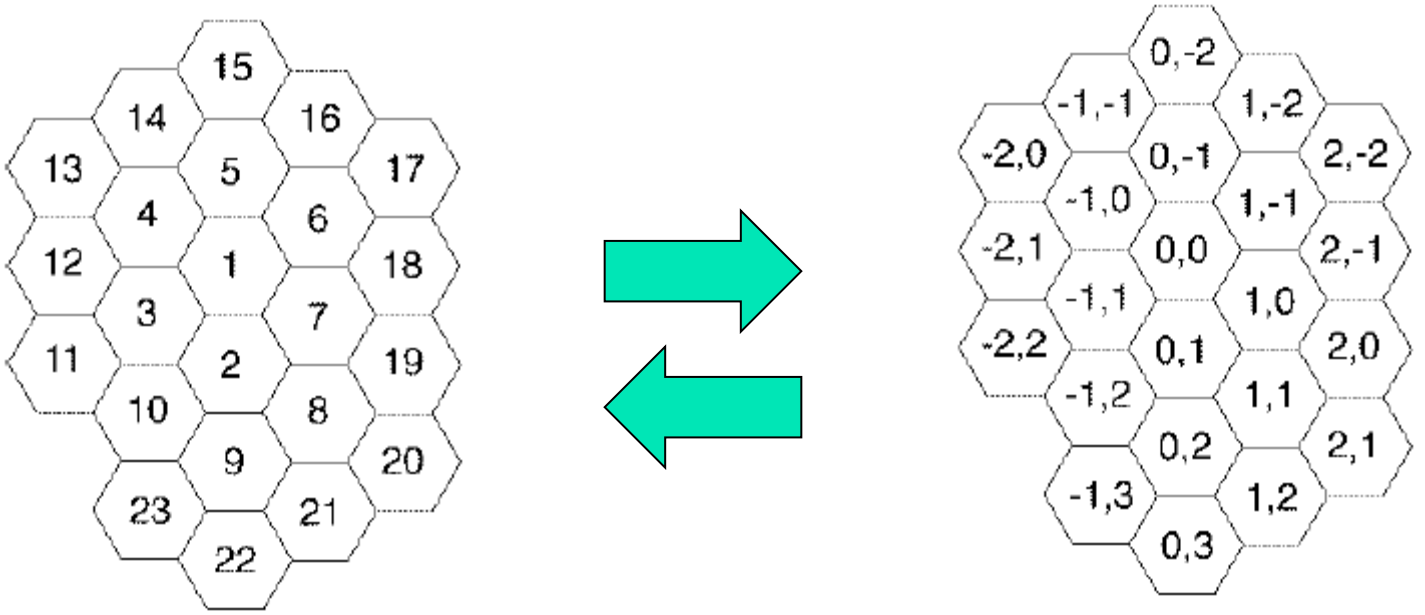
Sample Input

```
5 7 8 9 6 1 9 0 9 8 4 6
```

Sample Output

```
40 172
```

Problem 2: Convert coordinate numbers



Geometry

- You are supposed to know enough geometry
- How can you apply this knowledge to solve problems?
  - even a simple solution with pencil and paper needs severe elaboration for a program
- You should be familiar with two disciplines
  - geometry
  - representation of geometric problem/solution as a software program

Lines

- representations
  - two points: (x1, y1) and (x2, y2)
  - a single point and a slope: y = mx + b
    - $m = (y1 - y2)/(x1 - x2)$
    - what if (x1-x2) is 0? – needs special attention for division
  - more general: ax + by + c = 0

```
typedef struct {
    double a;
    double b; // default value is 1
    doublec;
} line
```

  - any one of the above can be converted to any other

Excercises

- find an intersection of two lines

$$l_1 : y = m_1 x + b_1 \text{ and } l_2 : y_2 = m_2 x + b_2$$

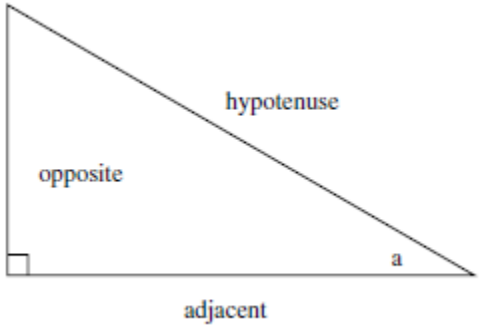
  - check if they are parallel
  - else find

$$x = \frac{b_2 - b_1}{m_1 - m_2}, \qquad y = m_1 \frac{b_2 - b_1}{m_1 - m_2} + b_1$$

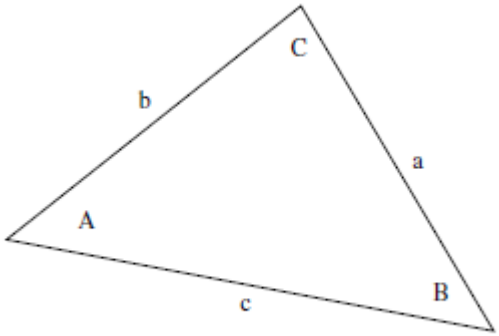
- what else?
  - angles of two lines
  - closest point on a line
  - rays – half line with an origin

Triangles and Trigonometry

- angles: use radians since most libraries use them
- make yourself comfortable with terminologies
  - right triangle
  - perpendicular lines
  - internal/external angle
  - equilateral
  - hypotenuse



- Remember them?



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

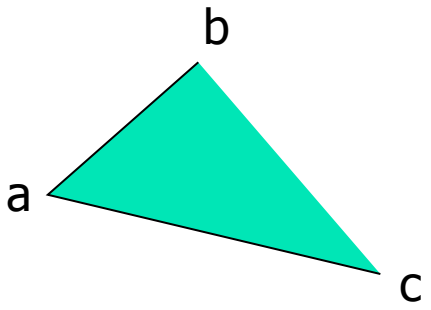
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Triangle Area

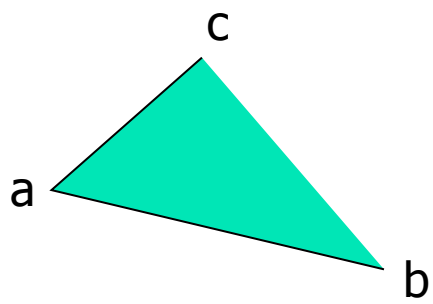
singed triangle area

$$2 \cdot A(T) = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} = a_x b_y - a_y b_x + a_y c_x - a_x c_y + b_x c_y - c_x b_y$$

positive



negative

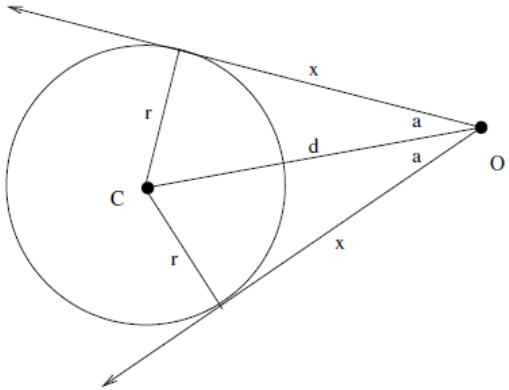


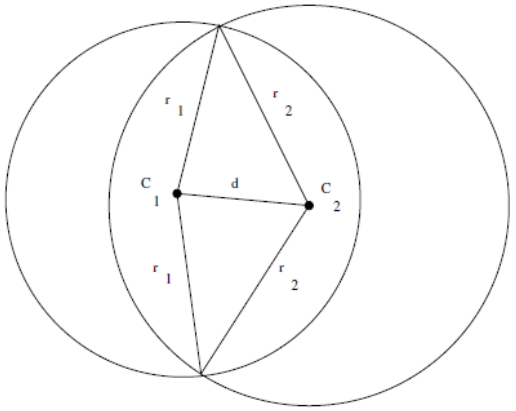
Circles

- representation

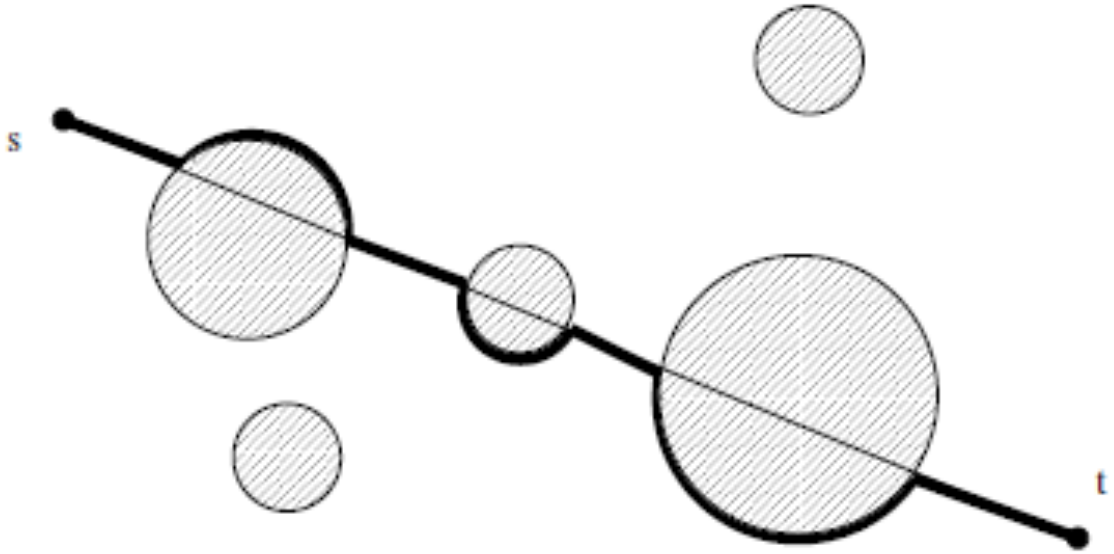
```
typedef struct {
    point c;        /* center */
    double r;       /* radius */
} circle;
```

- formula: $r^2 = (x - x_c)^2 + (y - y_c)^2$
- problems
  - tangent line
  - intersection points





Excercise



- find the travel distance
  1. find circles intersecting the s-t line
  2. find the length of the chord
  3. compute the arc length

Libraries

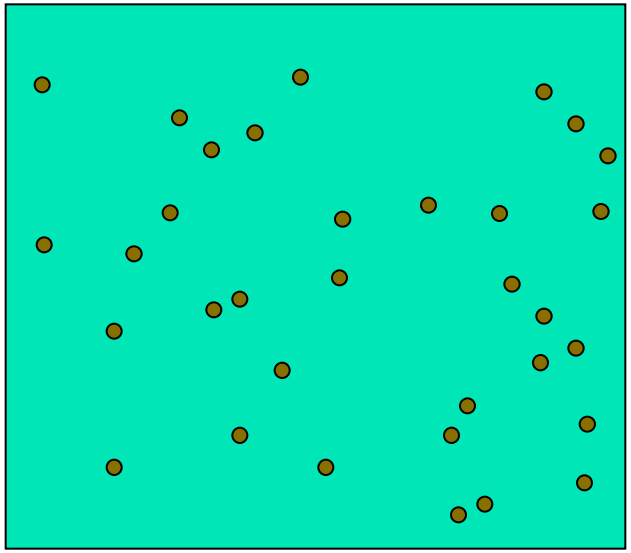
```
#include <math.h>

double cos(double x);     /* compute the cosine of x radians */
double acos(double x);    /* compute the arc cosine of [-1,1] */

double sin(double x);     /* compute the sine of x radians */
double asin(double x);    /* compute the arc sine of [-1,1] */

double tan(double x)      /* compute the tangent of x radians */
double atan(double x);    /* compute the principal arctan of x */
double atan2(double y, double x); /* compute the arc tan of y/x */
```

Excercise



- 50- cm square plate
- place a circle of 5 cm to contain most chips

Computational Geometry

- You need to handle geometry data
  - graphics
  - navigation
  - CAD
- Objects are usually
  - lines and polygones

- computational geometry deal with them using a computer
  - the most fascinating and growing area in mathematical sciences

Line Segments

- a portion of a line

```
typedef struct {
    point  p1, p2;
} segment;
```

- Are two segments intersect?
  - first, deal with degeneracy cases
    - parallel
    - total overlap

Polygons

- definition
  - a closed chain of non-intersecting line segments

```
typedef struct {
    int  n;                 /* number of vertices */
    point  p[MAXPOLY];      /* vertices  are ordered? */
} polygon;
```

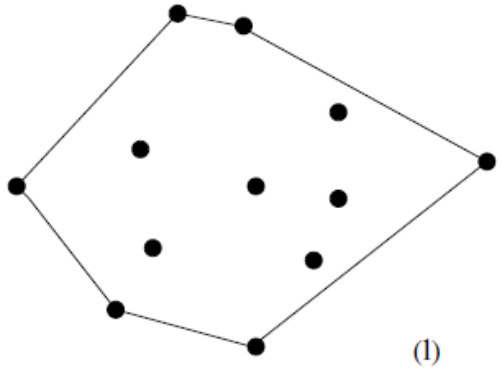
  - convex
    - all internal angles are acute
    - "singed triangle are function" tests if a point is left or right of a line

```
bool ccw(point a, point b, point c)
{
      double signed_triangle_area();

      return (signed_triangle_area(a,b,c) > EPSILON);
}
```

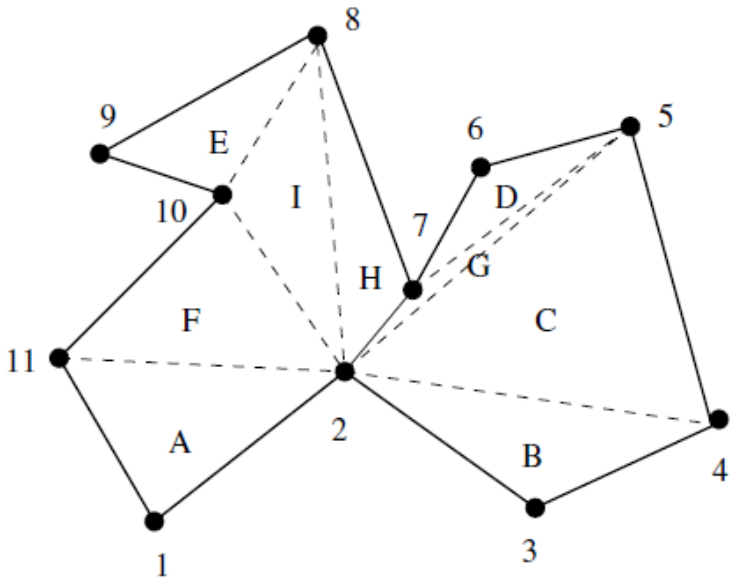
Convex Hulls

- similar to sorting
- The smallest polygon containing a set of points



- Solutions
  - find the leftmost-lowest point (origin)
  - sort points according to angles to the origin
  - add one by one
  - be careful of degeneracy

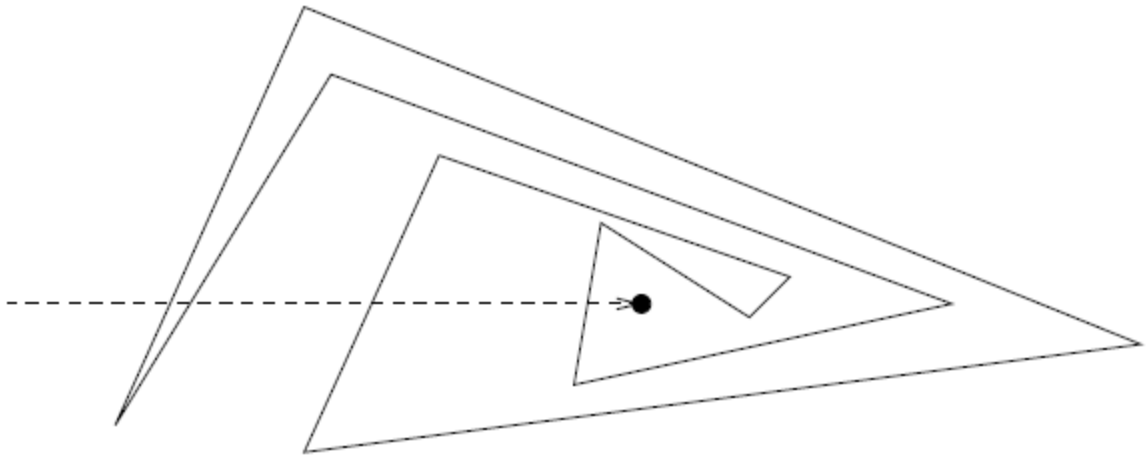
Finding area of a polygon



- triangulation
  - convex is easy
  - then, make it a convex
- ear cutting method
  - find an ear, calculate its area
  - do this until there remains a triangle

- How to test if a point is inside a polygon?
- What if a chord of the ear cuts other segment of the polygon?
  - test if any vertex is within an ear
  - better solution?

Jordan Curve Theorem



Lattice Polygon

$$A(P) = I(P) + B(P)/2 - 1$$

- A: area
- I: number of points inside the polygon
- B: number of points on the border
- How to computer I and B
  - B is easy
  - better solution for I?