Combinatorics

Combinatorics is a branch of mathematics concerning the study of finite or countable discrete structures

In the past, ad-hoc solutions were available.
Now, ...

Counting Techniques

- product rule
  - 5 shirts * 4 pants = total 20 combinations
- sum rule
  - if any one is missing from the closet, it is one of 9 clothing pieces
- Inclusion-Exclusion Formula
  - $|A \cup B| = |A| + |B| - |A \cap B|$
  - $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |B \cap C| + |A \cap B \cap B|$
  - eliminates double counting
- Bijection
  - map a member of a set to a member of target set that is to be counted

More counting techniques

- Permutation
  - an arrangement of *n* items where every item appears exactly once - $n! = \prod_{i=1}^{n} i$
  - arrange three letters word using a, b, c
    - abc, acb, bac, bca, cab, cba, - 6 words (3!)
  - what if letters are reused? (strings)
    - aaa, abb, … - ${}_n\prod_r = n^r$
- Subsets
  - there are 2^n subsets of *n* items
    - a, b, c, ab, bc, ac, abc, and an empty set – 8 subsets

Recurrence Relations

- Recursively defined structures
  - tree, list, WFF, …
  - divide conquer algorithms work on them
- recurrence relation
  - polynomials $a_n = a_{n-1} + 1, a_1=1 \rightarrow a_n = n$
  - exponential $a_n = 2a_{n-1}, a_1=2 \rightarrow a_n = 2^n$
  - factorial $a_n = na_{n-1}, a_1=1 \rightarrow a_n = n!$

Binomial Coefficient

- ${}_nC_k$ – choose k things out of n

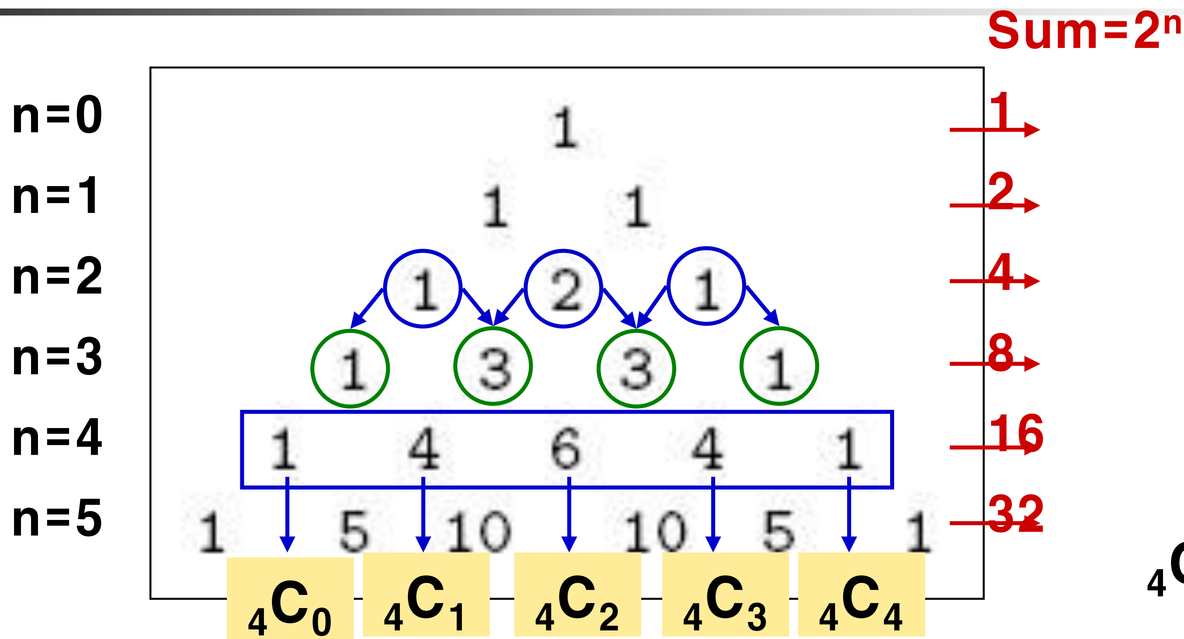
$$n!/((n-k)!k!)$$

- easy one
  - how many ways to form a k-member committee from n people
- Paths across a grid of n X m

Binomial Coefficient (2)

- coefficient of $(a+b)^n$
  - what is the coeff of $a^k b^{n-k}$?
  - how many ways to choose k a-terms out of n
  - $(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$
  - $(a + b)^3$ = aaa + 3aab + 3abb + bbb

Pascal's Triangle



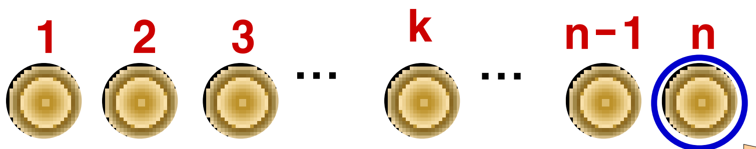
$_4C_3 = {}_3C_2 + {}_3C_3$

- prove that $_nC_k = {}_{(n-1)}C_{(k-1)} + {}_{(n-1)}C_k$

Binomial Coefficient(3)

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

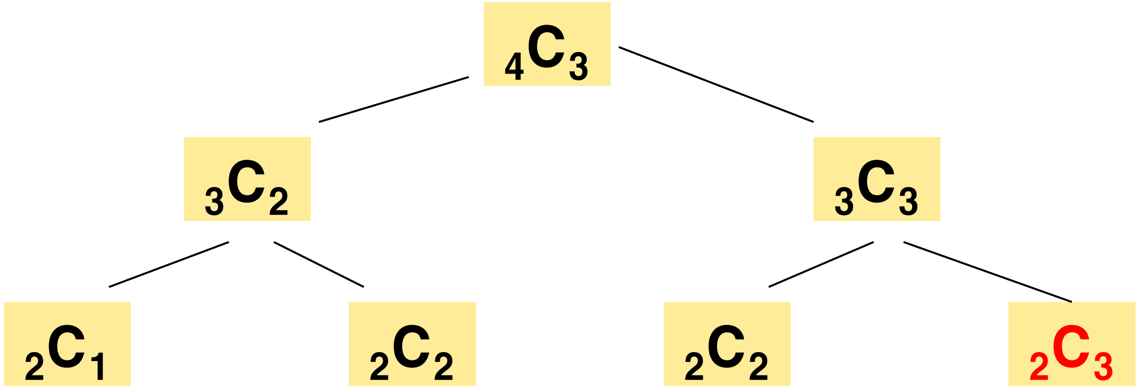
- $(n-k)!k!$ can cause overflow
- ${}_{n}C_{k} = {}_{(n-1)}C_{(k-1)} + {}_{(n-1)}C_{k}$



case1: this is included
same as choosing k-1 from n-1
case2: not
must choose k from n-1

Stop the recurrence

$${}_nC_k = {}_{(n-1)}C_{(k-1)} + {}_{(n-1)}C_k$$



- good ending points
  - ${}_nC_1 = n$
  - ${}_nC_n = 1$

```
int bc(int n, int k)
{
        if (k == 1) return(n);
        return  bc(n-1, k-1) + bc(n-1, k);
}
```

$_nC_k = {}_{(n-1)}C_{(k-1)} + {}_{(n-1)}C_k$

- safeness
  - what about $_3C_0$, $_2C_5$, $_{(-3)}C_5$
  - don't we need else part?
- time complexity
  - one comparison and one addition (let it be C1)
  - let recursion overhead be C2
  - $T(n, k) = T(n-1, k-1) + T(n-1, k) + C1 + C2$

Finonacci Numbers

$F_0 = 0$

$F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

```
int fb(int n)
{
        if (n < 0)  ????
        if (n < 2)) return(n);   // f(0) =0,  f(1) = 1
        else return fb(n-1) + fb(n-2);
}
```

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Eulerian Numbers

- the number of permutations of the numbers 1 to *n* in which exactly *m* elements are greater than the previous element

$$\left\langle {n \atop k} \right\rangle = k \left\langle {n-1 \atop k} \right\rangle + (n-k+1) \left\langle {n-1 \atop k-1} \right\rangle$$

- review Stirling Numbers, Partitions

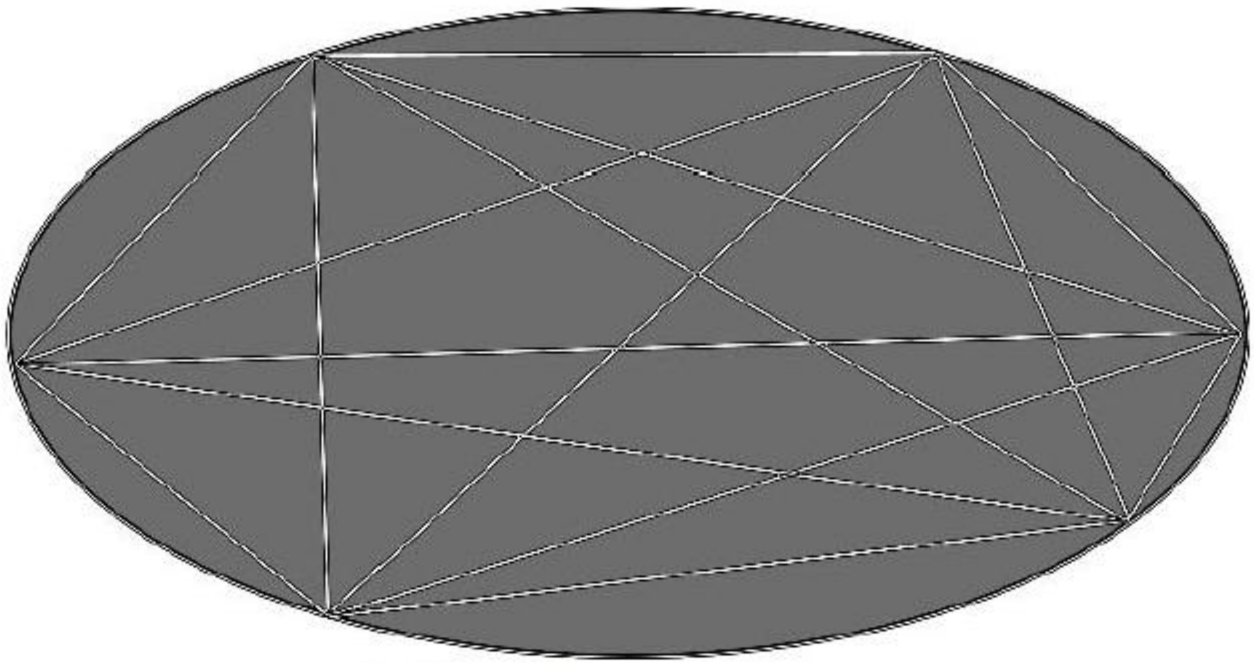
Induction

$T_n = 2T_{n-1} + 1, T_0=0$

n	0	1	2	3	4	5	6	7	...
T_n	0	1	3	7	15	31	63	127	...

prove/disprove that $T_n=2^n-1$

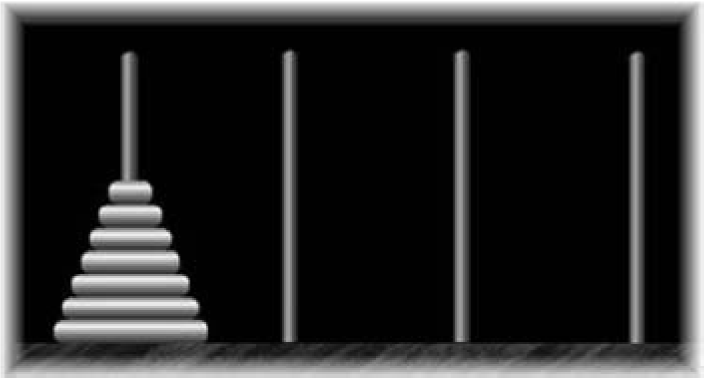
How Many Pieces of Land



Dividing the land when $n = 6$.

$$(i - 1) * i * (i * i - 5 * i + 18) / 24 + 1$$

Tower of Hanoi



The Four Needle (Peg) Tower of Hanoi