Lecture 5
Arithmetic and Algebra

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Numbers

- **Value ranges of a number data type**
  - Limited by the size of the data type
  - Usually 32bit or 64bit
  - Extremely large numbers

- **Arbitrary precision or arbitrarily big numbers**
  - Rarely we need
  - Data structures and arithmetic operations
Representation of Arbitrary Precision Integers

- **Arrays**

  ![Array Diagram]

  

- **Linked Lists**

  ![Linked List Diagram]
Bignum Data Type

- Sign and magnitude
- Reverse order

```c
#define MAXDIGITS 100 /* maximum length bignum */
#define PLUS 1 /* positive sign bit */
#define MINUS -1 /* negative sign bit */

typedef struct {
    char digits[MAXDIGITS]; /* represent the number */
    int signbit; /* PLUS or MINUS */
    int lastdigit; /* index of high-order digit */
} bignum;
```
In this section, we will implement the major arithmetic operations for the array-of-digits representation. Dynamic memory allocation and linked lists provide an illusion of being able to get unlimited amounts of memory on demand. However, linked structures can be wasteful of memory, since part of each node consists of links to other nodes. What dynamic memory really provides is the freedom to use space where you need it. If you wanted to create a large array of high-precision integers, a few of which were large and most of which were small, then you would be far better off with a list-of-digits representation, since you can't afford to allocate enormous amounts of space for all of them.

Our bignum data type is represented as follows:

```
#define MAXDIGITS 100 /* maximum length bignum */
#define PLUS 1 /* positive sign bit */
#define MINUS -1 /* negative sign bit */

typedef struct {
    char digits[MAXDIGITS]; /* represent the number */
    int signbit; /* PLUS or MINUS */
    int lastdigit; /* index of high-order digit */
} bignum;
```

Note that each digit (0–9) is represented using a single-byte character. Although it requires a little more care to manipulate such numbers, the space savings enables us to feel less guilty about not using linked structures. Assigning 1 and -1 to be the possible values of `signbit` will prove convenient, because we can multiply signbits and get the right answer.

Note that there is no real reason why we have to do our computations in base-10. In fact, using higher numerical bases is more efficient, by reducing the number of digits needed to represent each number. Still, it eases the conversion to and from a nice printed representation:

```
print_bignum(bignum *n)
{
    int i;

    if (n->signbit == MINUS) printf("- ");
    for (i=n->lastdigit; i>=0; i--)
        printf("%c","0' + n->digits[i]);

    printf("\n");
}
```
### Addition

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th>Addition</th>
<th>Subtraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>(+A) + (+B)</td>
<td>+(A+B)</td>
<td></td>
</tr>
<tr>
<td>(+A) + (-B)</td>
<td></td>
<td>+(A-B)</td>
</tr>
<tr>
<td>(-A) + (+B)</td>
<td></td>
<td>-(A-B)</td>
</tr>
<tr>
<td>(-A) + (-B)</td>
<td>-(A+B)</td>
<td></td>
</tr>
<tr>
<td>(+A) - (+B)</td>
<td></td>
<td>+(A-B)</td>
</tr>
<tr>
<td>(+A) - (-B)</td>
<td>+(A+B)</td>
<td></td>
</tr>
<tr>
<td>(-A) - (+B)</td>
<td></td>
<td>-(A+B)</td>
</tr>
<tr>
<td>(-A) - (-B)</td>
<td></td>
<td>-(A-B)</td>
</tr>
</tbody>
</table>
Addition

add_bignum(bignum *a, bignum *b, bignum *c)
{
    int carry;       /* carry digit */
    int i;           /* counter */
    initialize_bignum(c);

    if (a->signbit == b->signbit) c->signbit = a->signbit;
    else {
        if (a->signbit == MINUS) {
            a->signbit = PLUS;
            subtract_bignum(b,a,c);
            a->signbit = MINUS;
        } else {
            b->signbit = PLUS;
            subtract_bignum(a,b,c);
            b->signbit = MINUS;
        }
        return;
    }
    c->lastdigit = max(a->lastdigit,b->lastdigit)+1;
    carry = 0;

    for (i=0; i<=(c->lastdigit); i++) {
        c->digits[i] = (char)
                       (carry+a->digits[i]+b->digits[i]) % 10;
        carry = (carry + a->digits[i] + b->digits[i]) / 10;
    }
    zero_justify(c);
}
Zero Justification

zero_justify(bignum *n)  
{
    while ((n->lastdigit > 0) && (n->digits[ n->lastdigit ] == 0))
        n->lastdigit --;

    if ((n->lastdigit == 0) && (n->digits[0] == 0))
        n->signbit = PLUS;  /* hack to avoid -0 */
}

- Adjust lastdigit to avoid leading zeros
- Corrects -0
- Call after every arithmetic
Subtraction

Borrowing is tricky

Large no. – small no.

```c
subtract_bignum(bignum *a, bignum *b, bignum *c) {
    int borrow; /* anything borrowed? */
    int v; /* placeholder digit */
    int i; /* counter */
    if ((a->signbit == MINUS) || (b->signbit == MINUS)) {
        b->signbit = -1 * b->signbit;
        add_bignum(a,b,c);
        b->signbit = -1 * b->signbit;
        return;
    }
    if (compare_bignum(a,b) == PLUS) {
        subtract_bignum(b,a,c);
        c->signbit = MINUS;
        return;
    }
    c->lastdigit = max(a->lastdigit,b->lastdigit);
    borrow = 0;
    for (i=0; i<=(c->lastdigit); i++) {
        v = (a->digits[i] - borrow - b->digits[i]);
        if (a->digits[i] > 0)
            borrow = 0;
        if (v < 0) {
            v = v + 10;
            borrow = 1;
        }
        c->digits[i] = (char) v % 10;
    }
    zero_justify(c);
}
```
Comparison

```c
compare_bignum(bignum *a, bignum *b)
{
    int i;          /* counter */

    if ((a->signbit==MINUS) && (b->signbit==PLUS)) return(PLUS);
    if ((a->signbit==PLUS) && (b->signbit==MINUS)) return(MINUS);
    if (b->lastdigit > a->lastdigit) return (PLUS * a->signbit);
    if (a->lastdigit > b->lastdigit) return (MINUS * a->signbit);

    for (i = a->lastdigit; i>=0; i--) {
        if (a->digits[i] > b->digits[i])
            return(MINUS * a->signbit);
        if (b->digits[i] > a->digits[i])
            return(PLUS * a->signbit);
    }
    return(0);
}
```
### Multiplication

```c
multiply_bignum(bignum *a, bignum *b, bignum *c) {
    bignum row;                      /* represent shifted row */
    bignum tmp;                      /* placeholder bignum */
    int i, j;                        /* counters */

    initialize_bignum(c);

    row = *a;

    for (i=0; i<=b->lastdigit; i++) {
        for (j=1; j<=b->digits[i]; j++) {
            add_bignum(c,&row,&tmp);
            *c = tmp;
        }
        digit_shift(&row,1);
    }

    c->signbit = a->signbit * b->signbit;
    zero_justify(c);
}
```
 Digit Shift

digit_shift(bignum *n, int d)    /* multiply n by 10^d */
{
    int i;    /* counter */
    if ((n->lastdigit == 0) && (n->digits[0] == 0)) return;
    for (i=n->lastdigit; i>=0; i--)
        n->digits[i+d] = n->digits[i];
    for (i=0; i<d; i++) n->digits[i] = 0;
    n->lastdigit = n->lastdigit + d;
}
Exponentiation

- Repeated multiplication
- Reduce the number of multiplications
  \[ a^n = a^{n/2} \times a^{n/2} \times (a^{n\%2}) \]
By repeated subtractions

```c
divide_bignum(bignum *a, bignum *b, bignum *c)
{
    bignum row;                /* represent shifted row */
    bignum tmp;                /* placeholder bignum */
    int asign, bsign;          /* temporary signs */
    int i,j;                   /* counters */

    initialize_bignum(c);

    c->signbit = a->signbit * b->signbit;

    asign = a->signbit;
    bsign = b->signbit;

    a->signbit = PLUS;
    b->signbit = PLUS;

    initialize_bignum(&row);
    initialize_bignum(&tmp);

    c->lastdigit = a->lastdigit;
```
Division

for (i=a->lastdigit; i>=0; i--) {
    digit_shift(&row,1);
    row.digits[0] = a->digits[i];
    c->digits[i] = 0;
    while (compare_bignum(&row,b) != PLUS) {
        c->digits[i] ++;
        subtract_bignum(&row,b,&tmp);
        row = tmp;
    }
}

zero_justify(c);

a->signbit = asign;
b->signbit = bsign;

- Remainder of $a/b$ is $a-b(a/b)$
Base Conversion

- Converting base a number $x$ to base $b$ number $y$

- Two approaches
  - Left to right
    - MSD of $y$ is $d_i$ such that
      $$(d_i + 1)b^k > x \geq d_ib^k$$
    - $1 \leq d_i \leq b-1$
  - Right to left
    - LSD of $y$ is the remainder of $x$ divided by $b$
Polynomials

- Data structures
  - Array of coefficients
  - Linked list of coefficients

- Evaluation
  - Naïve approach
  - Horner’s Rule

\[((a_n x + a_{n-1})x + \ldots)x + a_0\]
The *reverse and add* function starts with a number, reverses its digits, and adds the reverse to the original. If the sum is not a palindrome (meaning it does not give the same number read from left to right and right to left), we repeat this procedure until it does.

For example, if we start with 195 as the initial number, we get 9,339 as the resulting palindrome after the fourth addition:

\[
\begin{array}{cccc}
195 & 786 & 1,473 & 5,214 \\
591 & 687 & 3,741 & 4,125 \\
+ & + & + & + \\
786 & 1,473 & 5,214 & 9,339 \\
\end{array}
\]

This method leads to palindromes in a few steps for almost all of the integers. But there are interesting exceptions. 196 is the first number for which no palindrome has been found. It has never been proven, however, that no such palindrome exists.

You must write a program that takes a given number and gives the resulting palindrome (if one exists) and the number of iterations/additions it took to find it.

You may assume that all the numbers used as test data will terminate in an answer with less than 1,000 iterations (additions), and yield a palindrome that is not greater than 4,294,967,295.
Reverse and Add

Input

The first line will contain an integer \( N \) \((0 < N \leq 100)\), giving the number of test cases, while the next \( N \) lines each contain a single integer \( P \) whose palindrome you are to compute.

Output

For each of the \( N \) integers, print a line giving the minimum number of iterations to find the palindrome, a single space, and then the resulting palindrome itself.

Sample Input

\[
3 \\
195 \\
265 \\
750
\]

Sample Output

\[
4 \ 9339 \\
5 \ 45254 \\
3 \ 6666
\]
Given any integer $0 \leq n \leq 10,000$ not divisible by 2 or 5, some multiple of $n$ is a number which in decimal notation is a sequence of 1’s. How many digits are in the smallest such multiple of $n$?

**Input**

A file of integers at one integer per line.

**Output**

Each output line gives the smallest integer $x > 0$ such that $p = \sum_{i=0}^{x-1} 1 \times 10^i$, where $a$ is the corresponding input integer, $p = a \times b$, and $b$ is an integer greater than zero.
## Ones

### Sample Input

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>9901</td>
<td></td>
</tr>
</tbody>
</table>

### Sample Output

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>
Stan and Ollie play the game of multiplication by multiplying an integer \( p \) by one of the numbers 2 to 9. Stan always starts with \( p = 1 \), does his multiplication, then Ollie multiplies the number, then Stan, and so on. Before a game starts, they draw an integer \( 1 < n < 4, 294, 967, 295 \) and the winner is whoever reaches \( p \geq n \) first.

**Input**

Each input line contains a single integer \( n \).

**Output**

For each line of input, output one line – either

Stan wins.

or

Ollie wins.

assuming that both of them play perfectly.
A Multiplication Game

Sample Input
162
17
34012226

Sample Output
Stan wins.
Ollie wins.
Stan wins.